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Abstract. We study the following problem: Given k paths that share the same vertex set, is there a simultaneous geometric embedding of these paths such that each individual drawing is monotone in some direction? We prove that for any dimension $d \geqslant 2$, there is a set of $d+1$ paths that does *not* admit a monotone simultaneous geometric embedding.

1 Introduction

Monotone drawings and simultaneous embeddings are well studied topics in graph drawing. Monotone drawings, introduced by Angelini et al. [2], are planar drawings of connected graphs such that, for every pair of vertices, there is a path between them that monotonically increases with respect to some direction. Monotone drawings of planar graphs have been studied both in the fixed and in the variable embedding settings and both with straight-line edges and with bends allowed along edges; recent papers on these topics include [3, 10, 12, 13].

The simultaneous (geometric) embedding problem was first described in a paper by Braß et al. [7]. The input is a set of planar graphs that share the same labeled vertex set (but the set of edges differs from one graph to another); the output is a mapping of the vertex set to a point set such that each graph admits a crossing-free drawing with the given mapping. The simultaneous embedding problem has also been studied by restricting/relaxing some geometric requirements; for example, while every pair of planar graphs sharing the same labeled vertex set admits a simultaneous embedding where each edge has at most two bends (see, e.g., [9, 11]), not even a tree and a path always admit a geometric simultaneous embedding (such that the edges are straight-line segments) [4]). See the book chapter on simultaneous embeddings by T. Bläsius et al. [6] for an extensive list of references on the problem and its variants.

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In this paper, we combine the two topics of simultaneous embeddings and monotone drawings. Namely, we are interested in computing geometric simultaneous embeddings of paths such that each path is monotone in some direction. Let $V = 1, 2, \ldots, n$ be a labeled set of vertices and let $\Pi = \{\pi_1, \pi_2, \ldots, \pi_k\}$ be a set of k distinct paths each having the same set V of vertices. We want to compute a labeled set of points $P = \{p_1, p_2, \ldots, p_n\}$ such that point p_i represents vertex i and for each path $\pi_i \in \Pi$ ($1 \leqslant i \leqslant k$) there exists some direction for which the drawing of π_i is monotone.

It is already known that any two paths on the same vertex set admit a monotone simultaneous geometric embedding in 2D, while there exist three paths on the same vertex set for which a simultaneous geometric embedding does not exist even if we drop the monotonicity requirement [7]. An example of three paths that do not have a monotone simultaneous geometric embedding in 2D can also be derived from a paper of Asinowski on suballowable sequences [5]. On the other hand, it is immediate to see that in 3D any number of paths sharing the same vertex set admits a simultaneous geometric embedding: Namely, by suitably placing the points in generic position (no 4 coplanar), the complete graph has a straight-line crossing-free drawing; however, the drawing of each path may not be monotone. This motivates the following question: Given a set of paths sharing the same vertex set, does the set admit a monotone simultaneous geometric embedding in d-dimensional space for $d \geqslant 3$?

Our main result is that for any dimension $d \geqslant 2$, there exists a set of $d+1$ paths that does not admit a monotone simultaneous geometric embedding in d-dimensional space. Our proof exploits the relationship between monotone simultaneous geometric embeddings in d-dimensional space and their corresponding representation in the dual space. Our approach extends to d dimensions the primal-dual technique described in a recent paper by Aichholzer et al. [1] on simultaneous embeddings of upward planar digraphs in 2D.

2 Definitions

Let $\vec{v}$ be a vector in $\mathbb{R}^d$ and let G be a directed acyclic graph with vertex set V. An embedding Γ of the vertex set V in $\mathbb{R}^d$ is called *$\vec{v}$-monotone* for G if the vectors in $\mathbb{R}^d$ corresponding to oriented edges of G have a positive scalar product with $\vec{v}$. Let $\mathcal{V} = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a set of $k > 1$ vectors in $\mathbb{R}^d$ and let $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ be a set of k distinct acyclic digraphs on the same vertex set V. A *$\mathcal{V}$-monotone simultaneous embedding* of $\mathcal{G}$ in $\mathbb{R}^d$ is an embedding Γ of V that is $\vec{v}_i$-monotone for G_i for any i. A *monotone simultaneous embedding* of $\mathcal{G}$ is a $\mathcal{V}$-monotone simultaneous embedding for some set $\mathcal{V}$ of vectors.

If a graph is a path on n (labeled) vertices, it can be trivially identified with a permutation of $[1, n]$. We look at the monotone simultaneous embedding problem in the dual space, by mapping points representing vertices to hyperplanes in $\mathbb{R}^d$. The dual formulation of monotone simultaneous embeddings is as follows (the equivalence of these formulations is shown in the next section). Let $\Pi = \{\pi_1, \pi_2, \ldots, \pi_k\}$ be a set of k permutations of $[1, n]$. A *parallel simultaneous*

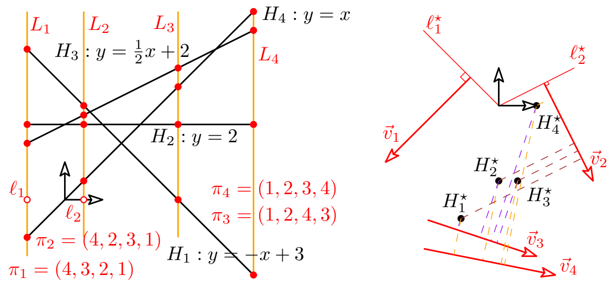

Fig. 1. Duality between monotone simultaneous embeddings and parallel simultaneous embeddings for $k = n = 4$ and $d = 2$.

embedding of Π in $\mathbb{R}^d$ is a set of n hyperplanes $H_1, H_2, \ldots, H_n$ and k vertical lines $L_1, L_2, \ldots, L_k$ such that the set of n points $L_j \cap H_{\pi_j(1)}, \ldots, L_j \cap H_{\pi_j(n)}$ is linearly ordered from bottom to top along L_j, for all j.

3 The Dual Problem and Non-Existence Results

The first two lemmas give duality results between monotone simultaneous embeddings and parallel simultaneous embeddings.

Lemma 1. *If a set of k permutations of $[1, n]$ admits a parallel simultaneous embedding in d dimensions, it also admits a monotone simultaneous embedding in d dimensions.*

Proof. Consider the following duality between points and hyperplanes, where we denote by $H^\star$ the dual of a non-vertical hyperplane H:

$$H : x_d = \left(\sum_{i=1}^{d-1} \alpha_i x_i\right) - \alpha_0, \qquad\qquad H^\star = (\alpha_1, \ldots, \alpha_{d-1}, \alpha_0).$$

This duality maps parallel hyperplanes to points that are vertically aligned (and vice-versa). Let $(H_i)_{1 \leqslant i \leqslant n}$, $(L_j)_{1 \leqslant j \leqslant k}$ be a parallel simultaneous embedding and refer to Fig. 1By definition, line L_j crosses hyperplanes $H_1, \ldots, H_n$ in the order $H_{\pi_j(1)}, H_{\pi_j(2)}, \ldots, H_{\pi_j(n)}$. The intersection points $L_j \cap H_{\pi_j(1)}, L_j \cap H_{\pi_j(2)}, \ldots, L_j \cap H_{\pi_j(n)}$ are collinear and therefore represent parallel hyperplanes in the dual plane. Consider the vector line $\vec{v}_j$ perpendicular to these hyperplanes and pointing downward. This line crosses them in the order $(L_j \cap H_{\pi_j(1)})^\star, (L_j \cap H_{\pi_j(2)})^\star, \ldots, (L_j \cap H_{\pi_j(n)})^\star$. Since point $H_i^\star$ lies in hyperplane $(L_j \cap H_i)^\star$, points ${H_i}^\star, 1 \leqslant i \leqslant n$, project on $\vec{v}_j$ in the order $H^\star_{\pi_j(1)}, H^\star_{\pi_j(2)}, \ldots, H^\star_{\pi_j(n)}$. Therefore $(H_i^\star)_{1 \leqslant i \leqslant n}$ is an embedding such that path π_j is $\vec{v}_j$-monotone, for all j. $\square$

Lemma 2. *If a set $(\pi_j)_{1 \leqslant j \leqslant k}$ of k permutations of $[1, n]$ admits a monotone simultaneous embedding in d dimensions, there is a set $(\pi'_j)_{1 \leqslant j \leqslant k}$ that admits a parallel simultaneous embedding in $\mathbb{R}^d$ where, for every j, π'_j is either equal to π_j or to its reverse.*

Proof. As in the proof of Lemma 1, we consider point-hyperplane duality. Let $(p_i)_{1 \leqslant i \leqslant n}$ be an embedding $\vec{v}_j$-monotone for π_j, and $(p_i^\star)_{1 \leqslant i \leqslant n}$ the corresponding set of dual hyperplanes. Let H_j be a hyperplane with normal vector $\vec{v}_j$, $1 \leqslant j \leqslant n$. Define L_j to be the vertical line through point $H_j^\star$. By construction, the points $\left(L_j \cap p^\star_{\pi_j(i)}\right)_i$ appear in order on L_j for one of the two possible orientations of L_j. In particular, when $\vec{v}_j$ points downward, L_j lists the points $L_j \cap p^\star_{\pi_j(i)}$ from bottom to top and vice versa. □

We now prove results of existence and non-existence of parallel simultaneous embeddings, starting with a very simple result of existence.

Proposition 1. *Any set of d permutations on n vertices admits a monotone simultaneous embedding and a parallel simultaneous embedding in d dimensions.*

Proof. Choose d points in general position in the hyperplane $x_d = 0$ and draw a vertical line through each of these points. For each vertical line, choose a permutation and place on the line n points numbered according to the permutation. Fit a hyperplane through all the points with the same number. By construction, this set of hyperplanes is a parallel simultaneous embedding. Going to the dual, by Lemma 1, gives a monotone simultaneous embedding. Alternatively, the monotone embedding can be seen directly by considering the rank in the i-th permutation as the i-th coordinate. □

We now turn our attention to non-existence. For proving that there exists $k = d + 1$ permutations that do not admit a parallel simultaneous embedding in d dimensions, observe that we can consider any generic placement of the d first lines L_j since all such placements are equivalent through affine transformations. We then construct permutations for n big enough that cannot be realized with any placement of L_{d+1}. Similarly, constructing $k = d+1$ permutations that cannot be realized even up to inversion, yields the non-existence of a monotone simultaneous embedding in d dimensions by Lemma 2. We start with dimension 2, then move to dimension 3 and only then, generalize our results to arbitrary dimension. Observe that 2D results also follow from [5, Lemma 1 & Prop. 8], but we still present our proofs as a warm up for higher dimensions.

Lemma 3. *There exists a set of* 3 *permutations on* $\{0, 1, 2\}$ *that does not admit a parallel simultaneous embedding in 2D.*

Proof. Let L_1 and L_2 be two vertical lines, H_1 and H_2 two other lines, and let $\tau_1 = (1, 2)$ and $\tau_2 = (2, 1)$ be two permutations of $\{1, 2\}$. As in Fig. 2-left, if L_1 is left of L_2 and the intersections of H_1 and H_2 with L_j are ordered according to τ_i, we can deduce that $H_1 \cap H_2$ is between L_1 and L_2. It follows that a vertical line crossing H_1 below H_2 is to the left of that intersection point and thus to the left of L_2. Similarly, a vertical line crossing H_1 above H_2 is to the right of L_1. If we now consider $\tau_1 = \tau_2 = (1, 2)$ we have that a vertical line crossing H_1 above H_2 is not between L_1 and L_2 (Fig. 2-center). Consider now $\pi_1 = (1, 0, 2)$, $\pi_2 = (2, 1, 0)$ and $\pi_3 = (0, 2, 1)$. Restricting the permutations to $\{1, 2\}$ gives that L_3 must be

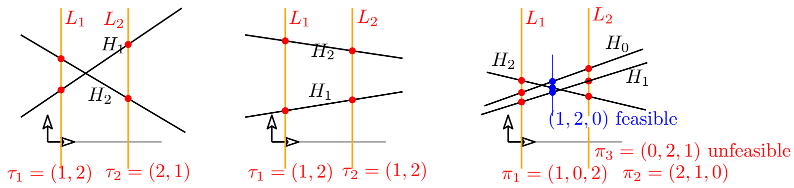

Fig. 2. Non-existence of two-dimensional parallel simultaneous embeddings.

right of L_1, restricting to $\{0,2\}$ gives that L_3 must be left of L_2, and restricting to $\{0,1\}$ gives that L_3 cannot be between L_1 and L_2 (Fig. 2-right). We deduce that no placement for L_3 can realize π_3. Notice that the reverse order $(1,2,0)$ can be realized and thus the dual of this construction is not a counterexample to simultaneous monotone embeddings. $\square$

Lemma 4. *There exists a set of* 3 *permutations on* 6 *vertices that does not admit a monotone simultaneous embedding in 2D.*

Proof. Let $\pi_1 = (f,b,d,e,a,c)$, $\pi_2 = (d,f,c,b,e,a)$, and $\pi_3 = (f,a,d,c,e,b)$. The sub-permutations of π_1, π_2 and π_3 on $\{a,b,c\}$ are (by matching (a,b,c) to $(0,1,2)$) the 3 permutations that do not admit a parallel simultaneous embedding in the proof of Lemma 3. The same is obtained by reversing only π_1 (resp. π_2, π_3) and considering sub-permutations on $\{a,c,d\}$ (resp. $\{d,b,e\}$, $\{b,f,d\}$). Other possibilities are symmetric and Lemma 2 yields the result. $\square$

Lemma 5. *There exists a set of* 4 *permutations on* 5 *vertices that does not admit a parallel simultaneous embedding in 3D.*

Proof. As in the proof of Lemma 1 we consider 3 points ℓ_1, ℓ_2, ℓ_3 in general position in the hyperplane $x_3 = 0$ and the 3 vertical lines L_1, L_2, L_3 going through these points. Let L be a vertical line (candidate position for L_4) and ℓ its intersection with $x_3 = 0$. We consider the 3 permutations $\tau_1 = (1,2,3)$, $\tau_2 = (2,3,1)$, $\tau_3 = (3,1,2)$ defining the vertical order of the intersections of L_1, L_2, L_3 with hyperplanes $(H_i)_{1 \leqslant i \leqslant 3}$. We denote by $h_{i,j}$ the projection of the line $H_i \cap H_j$, $1 \leqslant i \neq j \leqslant 3$, onto the plane $x_3 = 0$. Since the three planes H_i, $1 \leqslant i \leqslant 3$ meet in one point, the lines $h_{1,2}$, $h_{2,3}$ and $h_{1,3}$ meet at the projection of that point onto the plane $x_3 = 0$.

Refer to Fig. 3. For L to cut H_2 below H_1, ℓ must be in the half-plane limited by $h_{1,2}$ and containing ℓ_2, and, similarly, for L to cut H_3 below H_2, ℓ must be in the half-plane limited by $h_{2,3}$ and containing ℓ_3. Thus, ℓ must be in a wedge with apex $h_{1,2} \cap h_{2,3}$ (Fig. 3-left). Since $h_{1,2}$ separates ℓ_2 from ℓ_1 and ℓ_3, and $h_{2,3}$ separates ℓ_3 from ℓ_1 and ℓ_2, the union of all wedges, for all possible positions of $h_{1,2}$ and $h_{2,3}$, is the union, $\mathcal{R}$, of triangle $\ell_1\ell_2\ell_3$ and the half-plane limited by $\ell_2\ell_3$ and not containing ℓ_1 (Fig. 3-center). To summarize, if $\tau_1 = (1,2,3)$, $\tau_2 = (2,3,1)$, $\tau_3 = (3,1,2)$, and $\tau_4 = (3,2,1)$ then ℓ_4 (the intersection point of L_4 with the hyperplane $x_3 = 0$) must lie in this region $\mathcal{R}$.

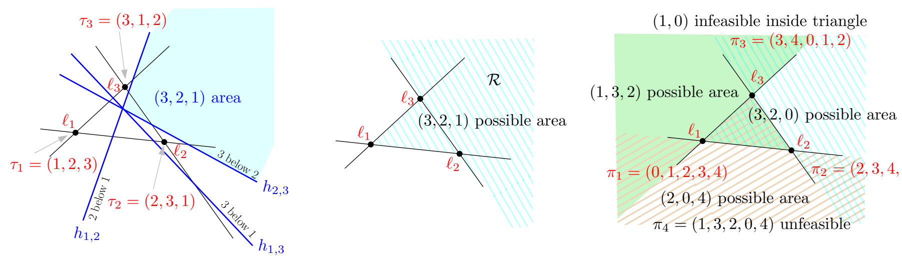

Fig. 3. Non-existence of 3D parallel simultaneous embeddings for 5 vertices.

Next, we build the permutations π_1, π_2, π_3 and π_4 by repeating this example as follows: $\pi_1 = (0,1,2,3,4)$, $\pi_2 = (2,3,4,0,1)$, $\pi_3 = (3,4,0,1,2)$, and $\pi_4 = (1,3,2,0,4)$. The restriction of these permutations to $\{0,2,3\}$ yields that ℓ_4 must be in the triangle or in the half-plane limited by $\ell_2\ell_3$ and not containing ℓ_1. The restriction to $\{1,2,3\}$ yields that ℓ_4 must be in the triangle or in the half-plane limited by $\ell_1\ell_3$ and not containing ℓ_2. The restriction to $\{0,2,4\}$ yields that ℓ_4 must be in the triangle or in the half-plane limited by $\ell_1\ell_2$ and not containing ℓ_3. Finally, considering $\{0,1\}$ yields that ℓ_4 must be outside the triangle (Fig. 3-right). Thus there is no possibility for placing L_4. $\square$

Lemma 6. *There exists a set of* 4 *permutations on* 40 *vertices that does not admit a monotone simultaneous embedding in 3D.*

Sketch of proof. The idea is to concatenate several versions of the counterexample of the previous lemma to cover all possibilities of reversing permutations. Note that the number of 40 vertices is not tight. $\square$

Lemma 7. *There exists a set of $d+1$ permutations on $3 \cdot 2^d$ vertices that does not admit a parallel simultaneous embedding in d dimensions.*

Sketch of proof. As in Lemma 5, the idea is to consider the simplex $(\ell_j)_{1 \leqslant j \leqslant d}$ and to construct the permutations for the L_i in order to prevent all possibilities for placing ℓ_{d+1}. $\square$

To get a result in the dual, the difficulty is that we have to prevent not only some permutations but also their reverse versions.

Theorem 1. *There exists a set of $d+1$ permutations on $3 \cdot 2^{2d}$ vertices that does not admit a monotone simultaneous embedding in d dimensions.*

Sketch of proof. As for Lemma 6 we concatenate several versions of previous counter-example to cover all possibilities of reversing permutations. $\square$

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A Proofs

Proof of Lemma 6. We consider

π_1=(0, 1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21, 22, 23, 24, 30, 31, 32, 33, 34, 40, 41, 42, 43, 44, 50, 51, 52, 53, 54, 60, 61, 62, 63, 64, 70, 71, 72, 73, 74),

π_2=(2, 3, 4, 0, 1, 12, 13, 14, 10, 11, 22, 23, 24, 20, 21, 32, 33, 34, 30, 31, 41, 40, 44, 43, 42, 51, 50, 54, 53, 52, 61, 60, 64, 63, 62, 71, 70, 74, 73, 72),

π_3=(3, 4, 0, 1, 2, 13, 14, 10, 11, 12, 22, 21, 20, 24, 23, 32, 31, 30, 34, 33, 43, 44, 40, 41, 42, 53, 54, 50, 51, 52, 62, 61, 60, 64, 63, 72, 71, 70, 74, 73), and

π_4=(1, 3, 2, 0, 4, 14, 10, 12, 13, 11, 21, 23, 22, 20, 24, 34, 30, 32, 33, 31, 41, 43, 42, 40, 44, 54, 50, 52, 53, 51, 61, 63, 62, 60, 64, 74, 70, 72, 73, 71)

The idea is that we have eight groups of vertices. Group $\{0, 1, 2, 3, 4\}$ restricts exactly to the example of Lemma 5 and prevents going from primal to dual without reversing any permutations. Group $\{10, 11, 12, 13, 14\}$ prevents going from primal to dual reversing exactly π_4. The other groups prevent all combinations of reversals that leave the first permutation fixed. In this example we prefer the simplicity of proof to optimizing the number of vertices. Counterexamples with less vertices can be easily obtained by sharing vertices between the different groups. □

Proof of Lemma 7. As in previous lemma, we generalize Lemma 3 without trying to optimize the number of vertices in the permutations. We consider d points $(\ell_j)_{1\leqslant j\leqslant d}$ in general position in the hyperplane $x_d = 0$ and the d vertical lines $(L_j)_{1\leqslant j\leqslant d}$ going through these points. Let L_{d+1} be a (variable) vertical line and ℓ_{d+1} its intersection with $x_d = 0$. In a similar manner as in two dimensions consider $\tau_1 = (1, 0, 2)$, $\tau_2 = (2, 1, 0)$, and $\tau_3 = (0, 2, 1)$ and $\Pi_1 \subset \{i \mid 1 \leqslant i \leqslant d\}$, $\Pi_2 = \{i \mid 1 \leqslant i \leqslant d\} \setminus \Pi_1$, and $\Pi_3 = \{d + 1\}$; then assume that τ_i is the order of hyperplanes H_0, H_1, H_2 along L_k for any $k \in \Pi_i$. In other words, above ℓ_k, we have for instance H_2 above H_1 for $k \in \Pi_1$ and the converse for $k \in \Pi_2 \cup \Pi_3$.

In projection, this means that $h_{1,2} = H_1 \cap H_2$ separates $(\ell_i)_{i\in\Pi_1}$ from $(\ell_i)_{i\in\Pi_2}$ and that ℓ_{d+1} is on the side of $(\ell_i)_{i\in\Pi_2}$. Thus, ℓ_{d+1} must be in the pink hatched part in Fig. 4. Considering $h_{0,2}$ yields similarly that ℓ_{d+1} must be in the blue hatched part, and consequently, there is a hyperplane through ℓ_{d+1} that separates $(\ell_i)_{i\in\Pi_1}$ from $(\ell_i)_{i\in\Pi_2}$.

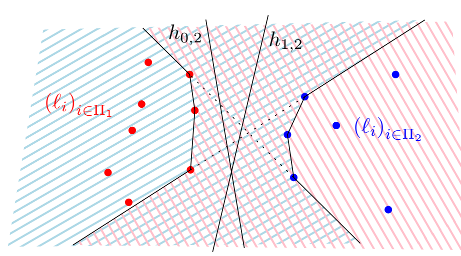

Fig. 4. Non-existence of a d-dimensional parallel simultaneous embedding.

Now we construct $\pi_1, \dots, \pi_{d+1}$ by concatenating one copy of τ_1, τ_2, and τ_3 with three new vertices for each possible partition of $\{i \mid 1 \leqslant i \leqslant d\}$ in Π_1 and Π_2. For any such partition, there is a hyperplane through ℓ_{d+1} that separates $(\ell_i)_{i\in\Pi_1}$ from $(\ell_i)_{i\in\Pi_2}$. Points $(\ell_j)_{1\leqslant j\leqslant d+1}$ can be seen in $\mathbb{R}^{d-1}$ (since $x_d = 0$) and considering the partition with $\Pi_1 = \emptyset$ yields that there is a hyperplane (in $\mathbb{R}^{d-1}$) through ℓ_{d+1} with all $(\ell_j)_{1\leqslant j\leqslant d}$ on one side. In other words, there is a hyperplane (in $\mathbb{R}^{d-1}$) separating ℓ_{d+1} from $(\ell_j)_{1\leqslant j\leqslant d}$. Projecting $(\ell_j)_{1\leqslant j\leqslant d}$ onto that plane (with a central projection with center ℓ_{d+1}) yields d points in $\mathbb{R}^{d-2}$, which can be partitioned in two sets, whose convex hulls intersect by Radon's theorem [14]. For this partition, there is no hyperplane through ℓ_{d+1} that separates $(\ell_i)_{i\in\Pi_1}$ from $(\ell_i)_{i\in\Pi_2}$, which is a contradiction. Hence, these $d+1$ permutations on $3 \cdot 2^d$ vertices prevent all placements for ℓ_{d+1}, which concludes the proof. (Note however that this number of vertices is clearly non-optimal.) $\square$

Proof of Theorem 1. A counterexample of $d+1$ permutations $(\pi_j)_{1\leqslant j\leqslant d}$ with no monotone simultaneous embedding must be a counterexample of $d+1$ permutations with no parallel simultaneous embedding for any set of permutations obtained from $(\pi_j)_{1\leqslant j\leqslant d}$ by reversing some of these permutations. Since there are 2^d ways of choosing which permutations are reversed, we can concatenate 2^d images of counterexamples from Lemma 7 by reversing some permutations so that the situation of Lemma 7 appears whatever choice of reversing is done. $\square$